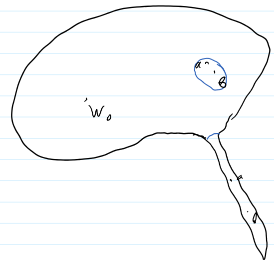
Tightness in the families of Loewner curves



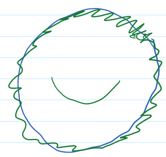
Setup: Ω – simply connected domain, $a, b \in \partial\hat{\Omega}$ – prime ends.

Metric on curves from a to b: (sup-distance)

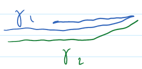
$$\mathrm{dist}(\gamma_1,\gamma_2) = \inf_{\substack{\text{parameterizations} \\ \text{of } \gamma_1 \text{ and } \gamma_2}} \sup \mathrm{dist}_{\Omega}(\gamma_1(t), \gamma_2(t)).$$

For locally-connected domains: equivalent to $\inf \sup |\gamma_1(t) - \gamma_2(t)|$.

Different from Hausdorff distance:



$$\mathrm{Hdist}(K_1, K_2) = \max\left(\sup_{y\in K_1} \mathrm{dist}(y, K_2), \sup_{y \in K_2} \mathrm{dist}(y, K_1)\right).$$



γ_1 and γ_2 are close in Hausdorff distance but not in sup-distance.

γ^δ – random curve = probability measure on curves. Usually comes from δ-lattice.

How to prove that $\gamma^\delta \to SLE_\kappa$?

1) Show tightness, i.e. precompactness in the weak* convergence of measures.

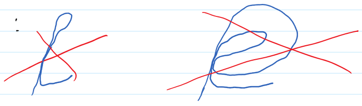
Consider subsequential limit $\gamma := \lim_{n\to\infty} \gamma^{\delta_n}$.

Enough to prove: $\gamma \overset{d}{=} SLE_\kappa$.

Remark: Tightness is: $\forall \varepsilon > 0\ \exists$ compact A in the space of curves such that $\forall \delta\ P(\gamma^\delta \in A) > 1 - \varepsilon$. (Prohorov Theorem).

Yuri Prohorov (1929-2013)

2) Prove that γ is supported on Löwner curves:



3) Consider Löwner driving function of γ and γ^δ (λ and λ^δ)

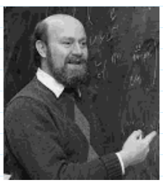
Using observable, show that $\lambda^\delta \to B(\kappa t)$.

One of the methods: consider asymptotic expansion of observables at ∞.

Another approach to proof (which gives rate of convergence).

1) Show that $\lambda^\delta(t) \simeq B(\kappa t)$ for small δ
   and $P\text{-}\lim \lambda^\delta(t) = B(\kappa t)$ – like in part 3).
2) Improve convergence to convergence of conformal images of interfaces in the model domain, i.e. $\varphi(\gamma^\delta) \to SLE_\kappa$
3) Promote to convergence of interfaces:
   $\gamma^\delta \to \gamma \sim SLE_\kappa$ in the domain

Not true in general that if $\lambda_n(t) \to \lambda(t)$ then
$\gamma_n(t) \to \gamma(t)$. But true with some
a-priori regularity!



Michael Aizenman

Almut Burchard

Aizenman-Burchard's framework for tightness:

Def Tortuosity

Let $\gamma : [0,1] \to \mathbb{C}$ be a curve, $\delta > 0$.

$M(\gamma, \delta) = \min \{ n : \underbrace{0 = t_0 < \dots < t_n = 1}_{\text{partition}},\ \sup_j \operatorname{diam} \gamma(t_{j-1}, t_j) \le \delta \}$.

Does not depend on parametrization.

In our settings, sometimes replace diam by Caratheodory diameter.

Another tortuisity measure:

$\widetilde{M}(\gamma, \delta) = \max \{ n : 0 = t_0 < t_1 < \dots < t_n = 1,\ \operatorname{dist}(\gamma(t_{j-1}), \gamma(t_j)) \ge \delta \}$.

Lemma $M(\gamma, 4\delta) \le \widetilde{M}(\gamma, \delta) \le \inf_{\varepsilon > 0} M(\gamma, \delta - \varepsilon)$

If $\gamma_n \to \gamma$ then $\varliminf M(\gamma_n, \delta) \ge M(\gamma, \delta)$, $\varlimsup \widetilde{M}(\gamma_n, \delta) \le \widetilde{M}(\gamma, \delta)$

Will be used to show the continuity of the rate of growth.

Proof. First inequality – because every segment of diam $\ge 4\delta$ contains a point of dist at least δ from both endpoints.
second – if $\operatorname{diam}(\gamma(t_{j-1}, t_j)) < \delta$ it can not have two points at distance $\ge \delta$.

Continuity properties: M – as defined as min,
$\widetilde{M}$ – as maximum.

Theorem (Uniform continuity and uniform tortuosity)

Let $\psi : (0,1] \to (0,1]$ be strictly increasing.

If γ has a parametrization $\gamma(t)$ with $\psi(|\gamma(t_1) - \gamma(t_2)|) \le |t_1 - t_2|$ for all $|\gamma(t_1) - \gamma(t_2)| \le 1$

then $\forall 0 < \delta \le 1$:

$$M(\gamma, \delta) \le \left\lceil \frac{1}{\psi(\delta)} \right\rceil.$$

Conversely, if $M(\gamma,\delta) \le \frac{1}{\psi(\delta)}$ then

$\exists$ parametrization $\gamma(t)$ which satisfies

$$\tilde{\psi}(|\gamma(t_1) - \gamma(t_2)|) \le |t_1 - t_2|,$$

$$\tilde{\psi}(\delta) := \frac{\psi(\delta/2)}{2(\log_2(4/\delta))^2}$$

Proof. Continuity $\Rightarrow$ tortuosity.

Cut the interior into $\left\lceil \frac{1}{\psi(\delta)} \right\rceil$ equal parts. Then for any t_1, t_2 in the same part,

$$\psi(|\gamma(t_1) - \gamma(t_2)|) \le |t_1 - t_2| \le \psi(\delta) \Rightarrow \operatorname{diam}(\gamma(t_j, t_{j+1})) \le \delta \blacksquare$$

Tortuosity $\Rightarrow$ continuity. We need to construct the right parametrization.

For a curve segment $\gamma(s,t)$, define

time of travel

$$t(s) := \frac{\sum_n (n+1)^{-2} \psi(2^{-n}) M(\gamma(0,s), 2^{-n})}{\sum_n (n+1)^{-2} \psi(2^{-n}) M(\gamma(0,1), 2^{-n})}$$

Parametrize the curve by $t(s)$.

Strictly increasing ($s_2 > s_1 \Rightarrow M(\gamma(0,s_1), 2^{-n}) < M(\gamma(0,s_2), 2^{-n})$ for large n)

Take $s_1 < s_2$. If $2^{-n} < |\gamma(s_1) - \gamma(s_2)|$ then

$$M(\gamma(0,s_2), 2^{-n}) - M(\gamma(0,s_1), 2^{-n}) \ge 1.$$

Thus
$$t(s_2) - t(s_1) \ge \frac{1}{2} \sum_{n : 2^{-n} < |\gamma(s_1) - \gamma(s_2)|} (n+1)^{-2} \psi(2^{-n}) \ge \frac{\psi(|\gamma(s_1) - \gamma(s_2)|/2)}{2(\log_2(4/|\gamma(s_1) - \gamma(s_2)|))^2} = \tilde{\psi}(|\gamma(s_1) - \gamma(s_2)|) \blacksquare$$

Corollary. Let $\tau(\gamma) := \inf\{\tau : \delta^{\tau} M(\gamma,\delta) \to 0\} = \lim_{\delta \to 0} \frac{\log M(\gamma,\delta)}{|\log \delta|}$

$\alpha(\gamma) := \sup\{\alpha : \gamma$ admits α-Hölder parametrization: $|\gamma(t_1) - \gamma(t_2)| \lesssim |t_1 - t_2|^{\alpha}\}$.

Then $\tau(\gamma) = \frac{1}{\alpha(\gamma)}$.

$$|t_1 - t_2| \gtrsim |\gamma(t_1) - \gamma(t_2)|^{\frac{1}{\alpha}}$$

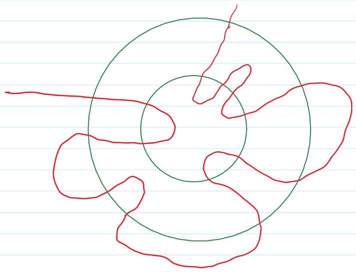
Proof Apply Theorem to $\psi(\delta) = \delta^{1/\alpha}$.

Let us now consider $A(z,r,R) := \{ w : r \le |w-z| \le R \}$
annulus.

AB condition for family of random curves γ_β.

$P(A(z,r,R)$ is traversed by k separate segments of $\gamma_\beta) \le K_k \left(\frac{r}{R}\right)^{\lambda(k)}$, $\beta < r < R$
for some $K_k < \infty$ and $\lambda(k) \to \infty$ as $k \to \infty$.

(even $\exists k : \lambda(k) > 2$)



6-arms

Theorem Let $(\gamma_\beta)_{\beta > 0}$ satisfy AB condition.

Then: 1) $\forall \varepsilon > 0$ all γ_β can be simultaneously parametrized by $\gamma_\beta(t)$:

$$|\gamma_\beta(t_1) - \gamma_\beta(t_2)| \le \underset{\text{random}}{K_{\varepsilon,\beta}} (\operatorname{diam} \gamma_\beta)^{-\frac{(1+\varepsilon)\lambda(1)}{2-\lambda(1)}} |t_1 - t_2|^{\frac{1}{2-\lambda(1)+\varepsilon}}$$

2) Tightness: $\exists \lim \gamma_{\beta_n} =: \gamma$ for some $\beta_n \to 0$.

Def γ has (ε, r_0, k) tempered crossing property if γ does not cross $A(z_0, r^{1+\varepsilon}, r)$ k or more times for any annulus $A(z_0, r^{1+\varepsilon}, r)$ with $r < r_0$.

Lemma Let $\gamma \subset B(0,R)$ for some R and has (ε, r_0, k) tempered crossing property. Then $M(\gamma, \delta) \le C k R^2 \delta^{-2(1+\varepsilon)}$, C is an absolute constant, $\delta \le r_0$.

Proof. Cover γ by $0 = t_0 < t_1 < \dots < t_n = 0$ where
$t_j := \inf \{ t > t_{j-1} : |\gamma(t) - \gamma(t_{j-1})| > \delta \}$
Then $M(\gamma, \delta) \le n$.
Cover γ by $\lesssim R^2 \delta^{-2(1+\varepsilon)}$ balls of diameter $\delta^{1+\varepsilon}$. If $B(z_0, \delta^{1+\varepsilon})$ is one of this disks then it contains at most k points $\gamma(t_j)$, since otherwise γ will cross $A(z_0, \delta^{1+\varepsilon}, \delta)$ more then k times.
So $M(\gamma, \delta) \le n \lesssim R^2 \delta^{-2(1+\varepsilon)} k$, as required ∎

Proof of 1):

... κ, as required ■

Proof of (1):

We just need to prove the tortuosity bound

$$M(\gamma,\delta) \le \kappa_{\varepsilon,\beta} \, (\operatorname{diam}\gamma)^{-\lambda(1)-\varepsilon} \, \delta^{-(2-\lambda(1)+\varepsilon)}$$

(By the relation between tortuosity and continuity)

Define random radius

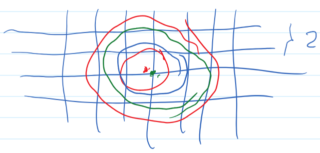
$$r_{\varepsilon,\beta,\kappa} := \inf\{ 0 < r \le 1 : \exists\, A(z, r^{1+\varepsilon}, r) \text{ traversed by} \ge \kappa \text{ segments of } \gamma_\beta \}.$$

(with $\inf \emptyset = 1$ now).

Claim: Let $\varepsilon\lambda(\kappa) > 2$. Then

$$P(r_{\varepsilon,\beta,\kappa} \le u) \le C(\varepsilon,\kappa)\, u^{\varepsilon\lambda(\kappa)-2}$$

Proof: Assume $\exists$ κ-crossing for some $r < u$.
Then $\exists$ κ-crossing of $A(z; 3\cdot 2^{-n(1+\varepsilon)}, 2^{-n-1})$, $z \in 2^{-n(1+\varepsilon)}\mathbb{Z}^2$ with $2^{-n} > r \ge 2^{-n-1}$.



So, by union bound,

$$P(\exists \kappa\text{-crossing on the scale } r \in (2^{-n-1}, 2^{-n}]) \le 2^{+2n(1+\varepsilon)} K_\kappa \left(\frac{2^{-n-1}}{3\cdot 2^{-n(1+\varepsilon)}} \right)^{\lambda(\kappa)} \le C(\kappa,\varepsilon)\, 2^{-n(-2\varepsilon+\varepsilon\lambda(\kappa))}$$

Now sum up for all $2^{-n-1} \le u$ to get the Claim ■

So if κ is as in Claim,

then $P(M(\gamma,\delta) > C\kappa(\operatorname{diam}\gamma)^2 \delta^{-2(1+\varepsilon)}) \le C\, \delta^{\varepsilon\lambda(\kappa)-2}$

tempered crossing for some $u \ge r_1$

Now we can use $\lambda(1)$ to estimate the diameter to arrive to the desired estimate.

Proof of tightness.

The set of curves with fixed Hölder bound C is compact, by Arzela-Ascoli.

By (1), for large C

$$P\left(|\gamma(t_1) - \gamma(t_2)| \le C\, (t_1 - t_2)^{\frac{1}{2-\lambda(1)+\varepsilon}} \right) > 1-\delta$$

which implies tightness ■

But will the limit be a Löwner curve?

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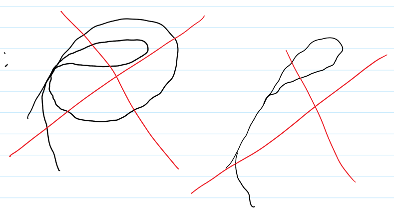
Problems:



Antti Kemppainen

Kemppainen-Smirnov Framework.

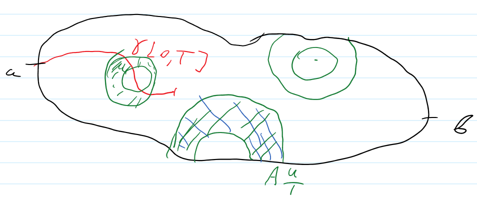
Def Let Ω - simply connected domain.

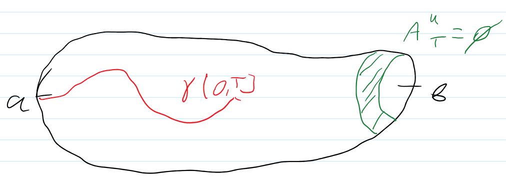
γ - random curve from $a \in \partial\Omega$ to $b \in \widehat{\partial\Omega}$ (prime ends)

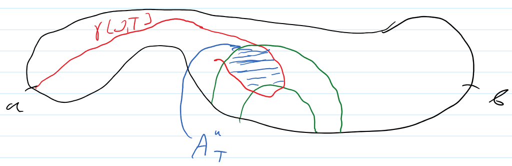
Let $A = A(z_0, r, R)$ - annulus. T - stopping time, Ω_T - connected component of b of $\Omega \setminus \gamma[0,T]$.

Avoidable set:

$$A_T^u := \begin{cases} \emptyset & \text{if } \partial B(z_0,r) \cap \partial\Omega_T = \emptyset \\ z \in \Omega_T \cap A: & \text{the connected component of } z \in \Omega_T \cap A \text{ does not disconnect } \gamma(T) \text{ from } b \text{ in } \Omega_T. \end{cases}$$







Def. $\gamma[T,1]$ makes an unforced crossing of $A = A(z_0, r, R)$ if it has a crossing contained in A^u_T.

KS condition: A family (γ^δ) satisfies KS-condition if

$\exists C>1, p<1$: $\forall T$ - stopping time, $\forall \delta$, $\forall A = A(z_0, r, R)$

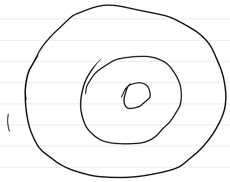
$0 < Cr \le R$,

$P(\gamma^\delta[T,1]$ makes an unforced crossing of $A \mid \gamma[0,T]) < p$

Equivalently: $\exists C>0, \alpha>0$:

$$P(\gamma^\delta[T,1] \text{ makes an unforced crossing of } A(z_0,r,R) \mid \gamma[0,T]) \le C\left(\frac{r}{R}\right)^\alpha.$$

Proof of equivalency:



Cut $\frac{R}{r}$ into rings $\sim \log \frac{R}{r}$ annuli with ratio C. Then to cross it = make $\sim \log \frac{R}{r}$ unforced crossings, with probability p

Theorem (Kemppainen - Smirnov)

Let γ^δ be a family of random curves satisfying KS-condition. Then

1) γ^δ is precompact and satisfy AB condition. (i.e. has uniform tortuosity bound)

2) If $\varphi: (\Omega, a, b) \to (\mathbb{H}, 0, \infty)$ - conformal, $\hat{\gamma}^\delta := \varphi(\gamma^\delta)$, and $\hat{K}_t$ - corresponding hulls in $\mathbb{H}$ then $a(t) := \mathrm{hcap}\, \hat{K}_t$ is strictly increasing, $\lim_{t\to\infty} a(t) = \infty$.

$\hat{\Omega}_t$ - unbounded component of $\mathbb{H} \setminus \hat{\gamma}[0,t]$.

$\hat{K}_t = \mathbb{H} \setminus \hat{\Omega}_t$.

3) If W^δ_t - driving process for $(\hat{K}_t)$, then W^δ_t is α-Hölder for any $\alpha < \frac{1}{2}$.

4) $E\left(\exp\left(\varepsilon \max_{s\le t} \frac{|W^\delta_s|}{\sqrt{t}}\right)\right) \le C$ for some $\varepsilon>0$, $C>0$ - depend only on KS parameters

5) Let (γ^δ) satisfies KS, (W^δ_t) - as above, driving functions. Then

$\gamma^{\delta_n} \to \gamma \iff (W^\delta_t) \to W_t$

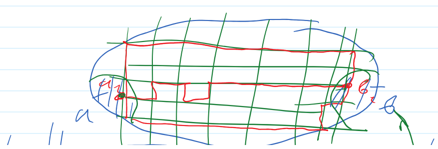
weakly wrt uniform convergence

weakly, uniform convergence on $[0,T]$ $\forall T>0$.

In particular, any weak limit is supported on Löwner curves.

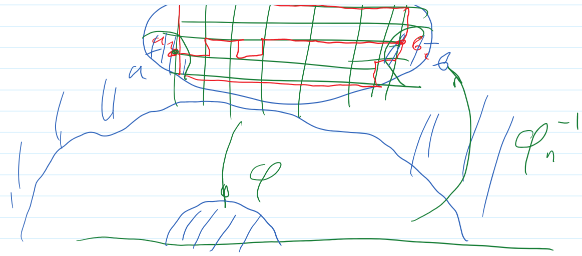
6) Let $(\Omega_n, a_n, b_n) \to (\Omega, a, b)$ in Caratheodory sense. Let $\varphi_n: (\Omega_n, a_n, b_n) \to (\mathbb{H}, 0, \infty)$ - conformal

$\varphi(\Omega, a, b) \to (\mathbb{H}, 0, \infty)$



Fix V_a, V_b - neighborhoods of a, b

sense. Let $\varphi_n : (\Omega_n, a_n, b_n) \to (\mathbb{H}, 0, \infty)$ — conformal
$\varphi\ (\Omega, a, b) \to (\mathbb{H}, 0, \infty)$



Fix V_a, V_b — neighborhoods of a, b,
$\hat{\Omega} := \Omega \setminus (V_a \cup V_b)$, $\hat{\Omega}_n := \varphi_n^{-1} \circ \varphi(\hat{\Omega}) \subset \Omega_n$.
Let γ_n from a_n to b_n satisfy KS uniformly
Then $\gamma_n | \hat{\Omega}_n$ has consequential weak sublimit
γ in $\hat{\Omega}$, satisfying KS condition.